INTERACTION BETWEEN SURFACE AND INTERNAL WAVES IN SHALLOW WATER

Analysis of data acquired at the U.S. Navy Electronics Laboratory Oceanographic Research Tower site

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PROBLEM

Investigate and report on the nature of internal waves of 2-to-20-minute periods at the U. S. Navy Electronics Laboratory (NEL) Oceanographic Research Tower. Specifically, study their relation to surface waves.

RESULTS

1. Resonant interaction between amplitude-modulated swell and internal waves may create internal waves of 2-to-20-minute periods. Internal waves with amplitudes of 1-3 meters can be produced within an interaction time of 15 minutes if the stratification of the water is such that

$$\left|\vec{k}_1 - \vec{k}_2\right|^2 / (\omega_1 - \omega_2)^2$$

is an eigenvalue of the internal wave equation ($\vec{k}_1$, $\vec{k}_2$, ω_1, and ω_2 being wave numbers and frequencies of the modulated swell).
2. Internal waves due to amplitude-modulated swell have the same characteristics as the modulation. Specifically, they have the same wavelength and period and travel in the same direction as the amplitude modulation.
3. The resonance process is most efficient in the case of a modulation which travels in the same direction as the carrier wave (the main constituent) of a swell. The creation of internal waves of this type is strongly dependent on the stratification of the water.
4. Preliminary evaluations of temperature and swell records from the NEL Tower show good agreement between periods of swell modulation and internal wave periods.
5. It is likely that internal waves are created on the entire continental shelf off Southern California during times of favorable stratification. This depends on tides and wind.

RECOMMENDATIONS

1. Verify, with additional measurements at the NEL Tower, the theory that internal waves are produced by swell.
2. Acquire, for complete analysis, long records of temperature fluctuations and surface waves in order to compute spectra with high resolution and statistical confidence.

3. Establish, by measurements of surface waves at two positions, and slick observations, that internal waves travel in the direction of the swell modulation.

ADMINISTRATIVE INFORMATION

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LIST OF SYMBOLS

If not defined in the article, the symbols have the following meaning:

A_n	Amplitude of W_n
$A^{\pm}_{mnr}$	Expansion coefficients, see equations (58) and (62)
a_m, a_n	See equations (63) and (64)
$F^{\pm}_{mn}$	See equation (57)
f_1, f_2	See equations (39) and (40)
$G^{\pm}_{mn}$	See equation (52)
g	Acceleration of gravity
H	Bottom depth
h	Horizontal components, used as an index
$\vec{k} \equiv (\kappa_1 \eta)$	Wave number vector
p	Pressure, $\bar{p}$ mean pressure
$\vec{u}$	(u, v, w) orbital current velocity of the waves, with components u, v, w in direction x, y, z
$W_n(z)$	Eigenfunction of n^{th} mode, according to equations (46) to (48)
$\vec{x}$	(x, y), horizontal plane
z	Vertical coordinate, pointing downward
$\Gamma(z)$	$\dfrac{1}{\bar{\rho}} \dfrac{d\bar{\rho}}{dz}$; Γ_0 = const
ζ	Amplitude of a wave
ρ	Density, $\bar{\rho}(z)$ mean density distribution
ϕ	Gravitational potential
ω	Angular frequency
∇	$\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$, del operator
$\nabla \cdot$	Divergence
∇^2	Laplace operator

INTRODUCTION

It was shown by E. C. LaFond[1-3] and O. S. Lee[4] that internal waves of 2-to-20-minute periods are a dominant feature in the thermal structure of the sea around the NEL Oceanographic Research Tower. From extensive measurements, E. C. LaFond derived that 50 percent of all waves had periods greater than 7.3 minutes and 50 percent of waves had heights of more than 5.6 feet (170 centimeters). Wave heights of more than 20 feet (6 meters) can sometimes be observed in water only 60 feet (18 meters) deep.

These short-period oscillations are only one part of the entire internal wave spectrum. They are superimposed on longer fluctuations of the mean thermocline, with changes mainly due to internal tides and wind.[5] Nevertheless, these waves in the 2-to-20-minute range are the most striking fluctuations besides the internal tides. Their occurrence seems to depend on several factors. There is neither a close relationship to the surface tides nor to the internal tides, but the changing stratification due to the tides seems to be of importance.

Most likely, these waves are long-crested, progressive waves traveling toward shore with a velocity of 20-to-40 feet per minute (10-to-20 centimeters per second). The measurements by O. S. Lee[4] indicate a beamwidth of only ±15 degrees. This agrees with former measurements by C. W. Ufford,[6] G. Ewing,[7] and E. C. LaFond.[1] C. S. Cox[8] got similar results. They are supported by observations of sea surface slicks, which are often closely related to internal waves. According to O. S. Lee,[4] the mean speed is 27 feet per minute (13.7 centimeters per second) and the direction 85 degrees.

Figure 1 gives an example of these waves measured on 4 October 1966, 1900-2000. The temperature fluctuations are shown for thermistors 3 to 21. The distance between the thermistors is 2.5 feet, thermistor 3 being 5 feet above the bottom and thermistor 21 about 20 feet below sea surface. The water depth is 60 feet.

After a calm period of several hours the waves start to occur at about 1900. The isotherm depth decreases during the next half hour and during that time high-amplitude internal waves are present. The period is not quite independent of depth. Shorter periods are generally observed near the surface than in deeper layers, but all waves are of first mode. This seems typical for the area.

The origin of these waves is rather obscure. There are no obvious meteorological or tidal forces that could produce the regular wave trains. If there were a constant coupling between the tidal phase and the occurrence of these waves, one would be inclined to interpret them as an adaptation of the changing mean stratification. But this is not possible. The only reason for their creation therefore seems to be surface waves.

F. K. Ball[9] has shown that, in the case of a two-layered model, resonance is possible for second order interactions between surface and internal boundary waves. S. A. Thorpe[10] extended the theory to wave interactions in a continuously stratified fluid. He showed that a transfer of energy from surface to internal waves may occur, and an internal wave generation mechanism will exist. The theory has been applied to situations which might be realized in the laboratory, but an application to natural conditions has not been attempted. For tank experiments (under somewhat extreme conditions) he found that the internal wave amplitude will be equal to that of the surface waves after an interaction time of only 28 seconds.

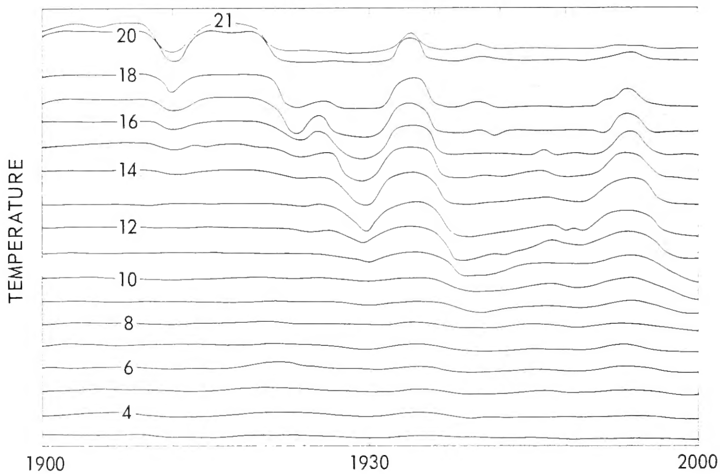

Figure 1. Temperature recorded by thermistors 3 to 21 at the NEL Tower (4 October 1966, 1900-2000). (Each curve represents the variation of the temperature from the mean for that numbered thermistor.)

K. E. Kenyon[11] extended the theory to a description of the entire internal wave spectrum which may result from the interaction of surface waves in the ocean. He gave a computation of scattering of swell energy into internal wave energy, using observations from the NEL Tower, from which he concluded that this process may be of minor importance for the creation of internal waves.

The following computations are similar to his, but we do not compute the energy spectra, and derive only the equations for the amplitudes of the waves. We come to different results, the reason for this being that Kenyon considered surface wave spectra only for the case that the directionality of the spectrum is given by $\cos^4\alpha$. This means that all surface waves are contained within an angle of about ±30 degrees. From this narrow directional swell spectrum, energy is fed to internal waves traveling mainly perpendicular to the swell. This, obviously, does not agree with the observations mentioned above. Both swell and internal waves travel mainly toward shore in the sea off Southern California.

Kenyon's conclusion that surface waves with a narrow directional spectrum do not feed considerable energy into the internal wave spectrum is supported by K. Hasselmann.[12] This agrees with our results, but it will be shown here that the process is about 100 times more effective for amplitude-modulated swell, which may be described by two surface waves traveling in directions which include an angle between 90 and 180 degrees. This still leads to a progressive wave, if the two amplitudes differ considerably in magnitude. If these two waves are given by

$$\zeta_1(\vec{x}, t) = A_1 \cos(\vec{k}_1 \cdot \vec{x} - \omega_1 t) \quad , \quad \zeta_2(\vec{x}, t) = A_2 \cos(\vec{k}_2 \cdot \vec{x} - \omega_2 t) \tag{1}$$

with $A_1 >> A_2$, the resulting wave can be described by

$$\zeta(\vec{x}, t) = \zeta_1(\vec{x}, t) + \zeta_2(\vec{x}, t) = A(\vec{x}, t) \cos(\vec{k}_1 \cdot \vec{x} - \omega_1 t - \phi(\vec{x}, t)) \tag{2}$$

where the amplitude $A(\vec{x}, t)$ and the phase $\phi(\vec{x}, t)$ are given according to

$$A(\vec{x},t) = \left\{ A_1{}^2 + A_2{}^2 + 2A_1A_2 \cos\left[(\vec{k}_1 - \vec{k}_2) \cdot \vec{x} - (\omega_1 - \omega_2)t\right] \right\}^{1/2} \tag{3}$$

and

$$\phi(\vec{x}, t) = \arctan \frac{A_2 \sin\left[(\vec{k}_1 - \vec{k}_2) \cdot \vec{x} - (\omega_1 - \omega_2)\, t\right]}{A_1 + A_2 \cos\left[(\vec{k}_1 - \vec{k}_2) \cdot \vec{x} - (\omega_1 - \omega_2)\, t\right]} \tag{4}$$

thus representing a swell which travels in a direction given by the wave vector $\vec{k}_1$ with an amplitude changing between $A_1 \pm A_2$ and a variable phase. Wave measurements at the NEL Tower indicate the reality of such a swell.

ENERGY TRANSFER FROM SURFACE WAVES TO INTERNAL WAVES BY RESONANCE

The process of resonant interaction is easy to comprehend, but the detailed analysis involves a great deal of algebra. A short outline of the main ideas may help in understanding the following computations.

The hydrodynamic equations are nonlinear. They include terms of the form $\vec{u} \cdot \nabla \vec{u}$, $\vec{u}$ being the velocity vector. In internal wave theory, these equations are linearized by perturbation methods. The result is a sum of mutually independent waves, each one conserving its own energy if friction is neglected. Many features are adequately described by this linear theory, but there exist circumstances under which the nonlinear terms can give rise to significant energy transfer between these primary wave solutions.

Surface waves are zero mode solutions of the internal wave equation. Internal waves may be of any mode larger than zero.

The computations in the following sections are based on this idea. Consider the simple nonlinear wave equation

$$\frac{\partial^2 \phi}{\partial t^2} + \omega^2 \phi - \phi^2 = 0 \tag{5}$$

Applying perturbation methods, $\phi = \phi^{(1)} + \phi^{(2)} + \cdots$, the first-order equation reads

$$\frac{\partial^2 \phi^{(1)}}{\partial t^2} + \omega^2 \phi^{(1)} = 0 \tag{6}$$

with the solution

$$\phi^{(1)} = \sum_n a_n e^{i\omega_n t} \tag{7}$$

These are mutually independent waves. The second-order equation is

$$\frac{\partial^2 \phi^{(2)}}{\partial t^2} + \omega^2 \phi^{(2)} = \phi^{(1)2} = \sum_m \sum_n a_m a_n e^{i(\omega_m + \omega_n)t} \tag{8}$$

The secondary waves $\phi^{(2)}$ are therefore forced waves with frequencies $\omega_m + \omega_n$. As long as these combination frequencies $\omega_m + \omega_n$ are not the same as a natural frequency of the system, the amplitudes of the secondary waves will remain small. However, when one of the combination frequencies $\omega_m + \omega_n$ is the same as one of the natural frequencies ω of the system, this wave never gets out of phase with the forcing wave, and a continuous energy transfer, only limited by the available energy in the primary waves, is possible. If this occurs, the solution of (8) is of the type $\phi^{(2)} \propto t$, that is, $\phi^{(2)}$ increases linearly with time. This is the resonance case. Its physical meaning is that energy from frequencies ω_n and ω_m is continuously fed into the frequency range $\omega_m + \omega_n$. We will study the problem of whether energy from surface waves in the frequency bands ω_m and ω_n can be transferred into internal waves in the frequency band $\omega_m \pm \omega_n$. Obviously, if ω_m and ω_n are swell frequencies, a transfer to the frequency band $\omega_m - \omega_n$ is possible only because internal waves have a cutoff at the Väisälä frequency, which is considerably lower than the swell frequencies. Additionally, the frequencies ω_m and ω_n must be very close together in order to feed energy into a frequency band corresponding to periods 2 to 20 minutes.

EQUATIONS AND BOUNDARY CONDITIONS

The hydrodynamic equations of a nonviscous, incompressible, stably stratified fluid in a Cartesian frame of reference with z pointing downward may be written as

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p - \rho \nabla \phi \tag{9}$$

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho = 0 \tag{10}$$

$$\nabla\cdot\vec{u} = 0 \tag{11}$$

The symbols have the following meaning: $\vec{u} \equiv (u, v, w)$ is the fluid velocity, p the pressure, ϕ the gravitational potential, and ρ the density. The effect of the earth's rotation is neglected. The boundary conditions are

$$w = -\left(\frac{\partial \zeta}{\partial t} + \vec{u}\cdot\nabla\zeta\right) \text{ at the sea surface, } z = -\zeta(\vec{x},t) \tag{12}$$

$$p = \text{const at the sea surface, } z = -\zeta(\vec{x}, t) \tag{13}$$

$$w = 0 \text{ at the bottom, } z = H = \text{const} \tag{14}$$

In order to apply the boundary conditions (12) and (13) at the undisturbed sea surface $z = 0$, we expand them about $z = 0$. The Taylor series of (12) is

$$w - \zeta\frac{\partial w}{\partial z} + \frac{\zeta^2}{2}\frac{\partial^2 w}{\partial z^2} + \cdots = -\left[\frac{\partial \zeta}{\partial t} + \left(\vec{u} - \zeta\frac{\partial \vec{u}}{\partial z} + \frac{\zeta^2}{2}\frac{\partial^2 \vec{u}}{\partial z^2} + \cdots\right)\cdot\nabla\zeta\right] \text{at } z = 0 \tag{15}$$

equation (13) yields

$$p - \zeta\frac{\partial p}{\partial z} + \frac{\zeta^2}{2}\frac{\partial^2 p}{\partial z^2} + \cdots = \text{const at } z = 0 \tag{16}$$

THE PERTURBATION EQUATIONS

Suppose the variables in (9) to (11) can be written as perturbation series

$$\vec{u} = \vec{u}^{(1)} + \vec{u}^{(2)} + \cdots, \quad p = p^{(0)} + p^{(1)} + p^{(2)} + \cdots, \quad \rho - \rho^{(0)} + \rho^{(1)} + \rho^{(2)} + \cdots,$$

$$\zeta = \zeta^{(1)} + \zeta^{(2)} + \cdots, \tag{17}$$

with $p^{(0)} = \bar{p}(z)$ and $\rho^{(0)} = \bar{\rho}(z)$, then, entering the equations (9) to (11) and (15) to (16), we arrive at the following equations, omitting terms of higher than second order.

0th Order Equations

$$\frac{d\bar{p}}{dz} = g\bar{\rho} \tag{18}$$

with

$$\bar{p} = \text{const at } z = 0. \tag{19}$$

1st Order Equations

$$\bar{\rho} \frac{\partial \vec{u}^{(1)}}{\partial t} + \nabla p^{(1)} + \rho^{(1)} \nabla \phi = 0 \tag{20}$$

$$\frac{\partial \rho^{(1)}}{\partial t} + w^{(1)} \frac{d\bar{\rho}}{dz} = 0 \tag{21}$$

$$\nabla \cdot \vec{u}^{(1)} = 0 \tag{22}$$

with

$$w^{(1)} = -\frac{\partial \zeta^{(1)}}{\partial t} \text{ at } z = 0 \tag{23}$$

$$p^{(1)} - \zeta^{(1)} \frac{d\bar{p}}{dz} = 0 \text{ at } z = 0 \tag{24}$$

$$w^{(1)} = 0 \text{ at } z = H \tag{25}$$

2nd Order Equations

$$\bar{\rho} \frac{\partial \vec{u}^{(2)}}{\partial t} + \nabla p^{(2)} + \rho^{(2)} \nabla \phi = -\left(\rho^{(1)} \frac{\partial \vec{u}^{(1)}}{\partial t} + \bar{\rho}\, \vec{u}^{(1)} \cdot \nabla \vec{u}^{(1)}\right) \tag{26}$$

$$\frac{\partial \rho^{(2)}}{\partial t} + w^{(2)} \frac{d\bar{\rho}}{dz} = -\vec{u}^{(1)} \cdot \nabla \rho^{(1)} \tag{27}$$

$$\nabla \cdot \vec{u}^{(2)} = 0 \tag{28}$$

with

$$w^{(2)} + \frac{\partial \zeta^{(2)}}{\partial t} = \zeta^{(1)} \frac{\partial w^{(1)}}{\partial z} - \vec{u}^{(1)} \cdot \nabla \zeta^{(1)} \text{ at } z = 0 \tag{29}$$

$$p^{(2)} - \zeta^{(2)} \frac{d\bar{p}}{dz} = \zeta^{(1)} \frac{\partial p^{(1)}}{\partial z} - \frac{\zeta^{(1)2}}{2} \frac{d^2\bar{p}}{dz^2} \text{ at } z = 0 \tag{30}$$

$$w^{(2)} = 0 \text{ at } z = H \tag{31}$$

The 0th order equations characterize the mean stable conditions; the 1st order ones describe linear internal and surface waves in that fluid; and the 2nd order equations give nonlinear effects of these waves, including energy transfers to other frequency bands. It is this set of equations which leads to energy flux from surface waves to internal waves.

THE EQUATIONS FOR $w(\vec{x}, z, t)$

We intend to eliminate u, v, ρ, and p in the left-hand sides of the 1st and 2nd order equations in order to get equations for w. The procedure is well-known.[13] For the 1st order equations we get

$$\nabla_h \cdot \left(\frac{\partial^2 w^{(1)}}{\partial t^2} + g\,\Gamma\, w^{(1)} \right) + \frac{\partial^4 w^{(1)}}{\partial t^2\, \partial z^2} + \Gamma \frac{\partial^3 w^{(1)}}{\partial t^2 \partial z} = 0 \tag{32}$$

where

$$\Gamma(z) = \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz}$$

Combining the boundary conditions (23) and (24) together with the horizontal components of (20) and the continuity equation (22), we find

$$\frac{\partial^3 w^{(1)}}{\partial t^2 \partial z} + g\, \nabla_h^{\,2}\, w^{(1)} = 0 \text{ at } z = 0 \tag{33}$$

and (25) remains the same

$$w^{(1)} = 0 \text{ at } z = H \tag{34}$$

Internal and surface waves of the first order are governed by equation (32) with boundary conditions (33) and (34). The problem has been solved analytically

for many density distributions $\bar{\rho}(z)$ and may be handled numerically for any distribution $\bar{\rho}(z)$. The result is a surface wave $w_0^{(1)}$ and a set of internal waves $\sum_{n=1}^{\infty} w_n^{(1)}$, each one being mutually independent and traveling with a phase velocity given by the eigenvalues of the problem.

The same procedure is used for deriving an equation for $w^{(2)}$ from (26) to (28): The operator $\nabla_h\cdot$ applied to (26) together with (28) yields

$$-\bar{\rho}\,\frac{\partial^2 w^{(2)}}{\partial t\partial z} + \nabla_h^2 p^{(2)} = -\nabla_h\cdot\left(\rho^{(1)}\,\frac{\partial \vec{u}_h^{(1)}}{\partial t} + \bar{\rho}\,\vec{u}^{(1)}\cdot\nabla\vec{u}_h^{(1)}\right) \tag{35}$$

Differentiating with respect to z and t we have

$$-\frac{d\bar{\rho}}{dz}\,\frac{\partial^3 w^{(2)}}{\partial t^2\partial z} - \bar{\rho}\,\frac{\partial^4 w^{(2)}}{\partial t^2\partial z^2} + \nabla_h^2\,\frac{\partial^2 p^{(2)}}{\partial z\partial t} = -\frac{\partial^2}{\partial z\partial t}\nabla_h\cdot\left(\rho^{(1)}\,\frac{\partial \vec{u}_h^{(1)}}{\partial t} + \bar{\rho}\vec{u}^{(1)}\cdot\nabla\vec{u}_h^{(1)}\right)$$

Combining the vertical component of (26) with (27) gives

$$\bar{\rho}\,\frac{\partial^2 w^{(2)}}{\partial t^2} + \frac{\partial^2 p^{(2)}}{\partial z\partial t} + g\,\frac{d\bar{\rho}}{dz}\,w^{(2)} = -g\vec{u}^{(1)}\cdot\nabla\rho^{(1)} - \frac{\partial}{\partial t}\left(\rho^{(1)}\,\frac{\partial w^{(1)}}{\partial t} + \bar{\rho}\vec{u}^{(1)}\cdot\nabla w^{(1)}\right),$$

which after applying ∇_h^2 and substituting $p^{(2)}$ in both equations above yields

$$\begin{aligned}\nabla_h^2\left(\frac{\partial^2 w^{(2)}}{\partial t^2} + g\Gamma w^{(2)}\right) + \frac{\partial^4 w^{(2)}}{\partial t^2\partial z^2} + \Gamma\,\frac{\partial^3 w^{(2)}}{\partial t^2\partial z} = -\frac{1}{\bar{\rho}}\;\nabla_h^2\Bigg\{ g\vec{u}^{(1)}\cdot\nabla\rho^{(1)} \\ + \frac{\partial}{\partial t}\left(\rho^{(1)}\,\frac{\partial w^{(1)}}{\partial t} + \bar{\rho}\vec{u}^{(1)}\cdot\nabla w^{(1)}\right)\Bigg\} + \frac{1}{\bar{\rho}}\,\frac{\partial^2}{\partial z\partial t}\,\nabla_h\cdot\left(\rho^{(1)}\,\frac{\partial \vec{u}_h^{(1)}}{\partial t} + \bar{\rho}\vec{u}^{(1)}\cdot\nabla\vec{u}_h^{(1)}\right)\end{aligned} \tag{36}$$

Combining the boundary conditions (29) and (23) together with (35) and eliminating $\frac{\partial p^{(1)}}{\partial z}$ in the inhomogeneous part by means of the vertical component of (26), we find

$$\begin{aligned}\frac{\partial^3 w^{(2)}}{\partial t^2\partial z} + g\nabla_h^2 w^{(2)} = \frac{\partial}{\partial t}\,\nabla_h\cdot\left(\frac{\rho^{(1)}}{\bar{\rho}}\,\frac{\partial \vec{u}^{(1)}}{\partial t} + \vec{u}^{(1)}\cdot\nabla\vec{u}_h^{(1)}\right) + \frac{\partial}{\partial t}\,\nabla_h^2\left[\left(\frac{g\rho^{(1)}}{\bar{\rho}} - \frac{\partial w^{(1)}}{\partial t}\right)\zeta^{(1)} \right. \\ \left. - \frac{g\Gamma\zeta^{(1)2}}{2}\right] + g\nabla_h^2\left(\zeta^{(1)}\,\frac{\partial w^{(1)}}{\partial z} - \vec{u}_h^{(1)}\cdot\nabla_h\zeta^{(1)}\right)\text{at } z = 0\end{aligned} \tag{37}$$

and (31) remains unaltered

$$w^{(2)} = 0 \text{ at } z = H \tag{38}$$

There are two driving forces responsible for second order internal waves:

(i) The body force f_1 in the fluid due to the nonlinear terms in the equations of motion and the incompressibility condition. It is given by the inhomogeneous part of (36)

$$f_1(\vec{x},z,t) = -\frac{1}{\bar{\rho}} \nabla_h^2 \left\{ g\vec{u}^{(1)} \cdot \nabla\rho^{(1)} + \frac{\partial}{\partial t}\left(\rho^{(1)} \frac{\partial w^{(1)}}{\partial t} + \bar{\rho}\vec{u}^{(1)} \cdot \nabla w^{(1)}\right)\right\}$$

$$+ \frac{1}{\bar{\rho}} \frac{\partial^2}{\partial z \partial t} \nabla_h \cdot \left(\rho^{(1)} \frac{\partial \vec{u}_h^{(1)}}{\partial t} + \bar{\rho}\vec{u}^{(1)} \cdot \nabla \vec{u}_h\right) \qquad (39)$$

(ii) The surface force f_2 due to the deviations of the sea surface from its mean level, given by the inhomogeneous part of (37)

$$f_2(\vec{x},z=0,t) = \frac{\partial}{\partial t} \nabla_h \cdot \left(\frac{\rho^{(1)}}{\bar{\rho}} \frac{\partial \vec{u}_h^{(1)}}{\partial t} + \vec{u}^{(1)} \cdot \nabla \vec{u}_h^{(1)}\right) + \frac{\partial}{\partial t} \nabla_h^2 \left[\left(\frac{g\rho^{(1)}}{\bar{\rho}} - \frac{\partial w^{(1)}}{\partial t}\right)\zeta^{(1)}\right.$$

$$\left. - \frac{g\Gamma\zeta^{(1)2}}{2}\right] + g\nabla_h^2 \left(\zeta^{(1)} \frac{\partial w^{(1)}}{\partial z} - \vec{u}_h^{(1)} \cdot \nabla_h \zeta^{(1)}\right) \qquad (40)$$

Comparing these equations with the simple example given at the beginning, the problem (9) to (14) corresponds to equation (5), and (32) to (34) corresponds to (6), and (36) to (38) corresponds to (8).

THE PRIMARY WAVE FIELD AND THE FORCES f_1 AND f_2

Suppose that the primary waves are progressive waves traveling in a direction given by the wave number vector $\vec{k} \equiv (\kappa, \eta)$. We will use the notation $k \equiv |\vec{k}| = \sqrt{\kappa^2 + \eta^2}$. A set of solutions of (32) to (34) is

$$\vec{u}_h^{(1)}(\vec{x},z,t) = -\sum_{n=1}^{\infty} A_n \frac{\vec{k}_n}{k_n^2} \frac{dW_n}{dz} \sin(\vec{k}_n \cdot \vec{x} - \omega_n t) \qquad (41)$$

$$w^{(1)}(\vec{x},z,t) = \sum_{n=1}^{\infty} A_n W_n \cos(\vec{k}_n \cdot \vec{x} - \omega_n t) \qquad (42)$$

$$\rho^{(1)}(\vec{x},z,t) = \frac{d\bar{\rho}}{dz} \sum_{n=1}^{\infty} \frac{A_n}{\omega_n} W_n \sin(\vec{k}_n \cdot \vec{x} - \omega_n t) \qquad (43)$$

$$p^{(1)}(\vec{x},z,t) = -\bar{\rho} \sum_{n=1}^{\infty} A_n \frac{\omega_n}{k_n^{\,2}} \frac{dW_n}{dz} \sin(\vec{k}_n \cdot \vec{x} - \omega_n t) \tag{44}$$

$$\zeta^{(1)}(\vec{x},z,t) = -\frac{1}{g} \sum_{n=1}^{\infty} A_n \frac{\omega_n}{k_n^{\,2}} \left.\frac{dW_n}{dz}\right|_{z=0} \sin(\vec{k}_n \cdot \vec{x} - \omega_n t) \tag{45}$$

where $W_n(z)$ is governed by

$$\frac{d^2 W_n}{dz^2} + \Gamma \frac{dW_n}{dz} + (g\Gamma - \omega_n^{\,2}) \frac{k_n^{\,2}}{\omega_n^{\,2}} W_n = 0 \tag{46}$$

with

$$\frac{dW_n}{dz} + \frac{g k_n^{\,2}}{\omega_n^{\,2}} W_n = 0 \quad \text{at } z = 0 \tag{47}$$

and

$$W_n = 0 \text{ at } z = H \tag{48}$$

and represents either surface or internal waves. From (41) to (45) we can derive the driving functions f_1 and f_2 according to (39) and (40). This includes a great deal of algebra. The final result is

$$f_1(\vec{x},z,t) = -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big\{ F^{+}_{mn}(z) \sin\left[(\vec{k}_m + \vec{k}_n)\cdot\vec{x} - (\omega_m + \omega_n)t\right]$$

$$+ F^{-}_{mn}(z) \sin\left[(\vec{k}_m - \vec{k}_n)\cdot\vec{x} - (\omega_m - \omega_n)t\right] \Big\} \tag{49}$$

$$f_2(\vec{x},0,t) = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big\{ G^{+}_{mn}(0) \sin\left[(\vec{k}_m + \vec{k}_n)\cdot\vec{x} - (\omega_m + \omega_n)t\right]$$

$$+ G^{-}_{mn}(0) \sin\left[(\vec{k}_m - \vec{k}_n)\cdot\vec{x} - (\omega_m - \omega_n)t\right] \Big\} \tag{50}$$

where $F^{\pm}_{mn}(z)$ and $G^{\pm}_{mn}(0)$ are given by

$$F^{\pm}_{mn}(z) = \Bigg\{ \frac{\vec{k}_m \cdot \vec{k}_n}{k_m^{\,2}\,\omega_n} \left[g\Gamma \pm \omega_n(\omega_m \pm \omega_n) \right] \frac{dW_m}{dz} W_n \mp \left[g\Gamma \pm \omega_n(\omega_m \pm \omega_n) \right] \frac{1}{\omega_n} W_m \frac{dW_n}{dz}$$

$$\mp \left[\frac{g}{\omega_n} \frac{1}{\bar{\rho}} \frac{d^2\bar{\rho}}{dz^2} - \frac{\omega_n(\omega_m \pm \omega_n)}{\omega_m} \Gamma \right] W_m W_n \Bigg\} |\vec{k}_m \pm \vec{k}_n|^2$$

$$+ \Bigg\{ \mp \frac{\vec{k}_m \cdot \vec{k}_n}{k_m^{\,2}} \frac{1}{\bar{\rho}} \frac{d}{dz} \left(\bar{\rho} \frac{dW_m}{dz} \frac{dW_n}{dz} \right)$$

$$+\frac{1}{\bar{\rho}}\frac{d}{dz}\left(\bar{\rho}W_m\frac{d^2W_n}{dz^2}\right)\mp\frac{\omega_n}{\omega_m}\frac{1}{\bar{\rho}}\frac{d}{dz}\left(\frac{d\bar{\rho}}{dz}W_m\frac{dW_n}{dz}\right)\Bigg\}$$

$$\left(1\pm\frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\right)(\omega_m\pm\omega_n) \tag{51}$$

$$\begin{aligned}G^{\pm}_{mn}(0)=\Bigg\{&\left[\pm\frac{\omega_r}{\omega_m}\Gamma W_m\frac{dW_n}{dz}\pm\frac{\vec{k}_m\cdot\vec{k}_n}{k_m^{\,2}}\frac{dW_m}{dz}\frac{dW_n}{dz}-W_m\frac{d^2W_n}{dz^2}\right]\left(1\pm\frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\right)(\omega_m\pm\omega_n)\\&\mp\left(\frac{g\Gamma-\omega_m^{\,2}}{\omega_m}W_m+\frac{\Gamma}{2}\frac{\omega_m}{k_m^{\,2}}\frac{dW_m}{dz}\right)\frac{\omega_n}{gk_m^{\,2}}\frac{dW_n}{dz}\left|\vec{k}_m\pm\vec{k}_n\right|^2(\omega_m\pm\omega_n)\\&+\left(\frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\omega_n+\omega_m\right)\frac{1}{k_m^{\,2}}\frac{dW_m}{dz}\frac{dW_n}{dz}\left|\vec{k}_m\pm\vec{k}_n\right|^2\Bigg\}_{z=0}\end{aligned} \tag{52}$$

THE SECONDARY WAVE FIELD IN THE RESONANCE CASE

The secondary wave field is given by (36) to (38), which with (49) and (50) read

$$\nabla_h^2\left(\frac{\partial^2w^{(2)}}{\partial t^2}+g\Gamma w^{(2)}\right)+\frac{\partial^4w^{(2)}}{\partial z^2\partial t^2}+\Gamma\frac{\partial^3w^{(2)}}{\partial t^2\partial z}=-\frac{1}{2}\sum_m\sum_n F^{\pm}_{mn}(z)\sin\left[(\vec{k}_m\pm\vec{k}_n)\cdot\vec{x}-(\omega_m\pm\omega_n)t\right] \tag{53}$$

$$\frac{\partial^3w^{(2)}}{\partial t^2\partial z}+g\,\nabla_h^2\,w^{(2)}=\frac{1}{2}\sum_m\sum_n G^{\pm}_{mn}(0)\sin\left[(\vec{k}_m\pm\vec{k}_n)\cdot\vec{x}-(\omega_m\pm\omega_n)t\right] \tag{54}$$

at $z=0$

$$w^{(2)}=0 \text{ at } z=H \tag{55}$$

where either the plus or minus sign holds. If eigenvalues of (53) exist, which fulfill the condition

$$\nu_r=\frac{\left|\vec{k}_m\pm\vec{k}_n\right|^2}{(\omega_m\pm\omega_n)^2} \tag{56}$$

the solution of (53) to (55) is

$$w^{(2)} = \sum_{mnr} \left\{ \left[\frac{(\omega_m \pm \omega_n)\, A^{\pm}_{mnr} W^{\pm}_r(z)}{4g\,|\vec{k}_m \pm \vec{k}_n|^2} + \widetilde{W}^{\pm}_r(z) \right] t \ \cos\left[(\vec{k}_m \pm \vec{k}_n)\cdot\vec{x} - (\omega_m \pm \omega_n)\, t \right] \right.$$
$$\left. + \mathring{W}^{\pm}_r(z) \sin\left[(\vec{k}_m \pm \vec{k}_n)\cdot\vec{x} - (\omega_m \pm \omega_n) t \right] \right\} \tag{57}$$

where $F^{\pm}_{mn}(z)$ has been expanded in an eigenfunction series according to

$$\frac{F^{\pm}_{mn}(z)}{\Gamma(z)} = \sum_r A^{\pm}_{mnr}\, W^{\pm}_r(z) \tag{58}$$

and the functions $W^{\pm}_r(z)$, $\widetilde{W}^{\pm}_r(z)$, and $\mathring{W}^{\pm}_r(z)$ are governed by the following equations

$$\left.\begin{aligned}
&\frac{d^2 W^{\pm}_r}{dz^2} + \Gamma \frac{dW^{\pm}_r}{dz} + \frac{\left[g\Gamma - (\omega_m \pm \omega_n)^2\right] |\vec{k}_m \pm \vec{k}_n|^2}{(\omega_m \pm \omega_n)^2} W^{\pm}_r = 0 \\
&\frac{dW^{\pm}_r}{dz} + \frac{g\,|\vec{k}_m \pm \vec{k}_n|^2}{(\omega_m \pm \omega_n)^2} W^{\pm}_r = 0 \text{ at } z = 0 \\
&W^{\pm}_r = 0 \text{ at } z = H
\end{aligned}\right\} \tag{59}$$

$$\left.\begin{aligned}
&\frac{d^2 \widetilde{W}^{\pm}_r}{dz^2} + \Gamma \frac{d\widetilde{W}^{\pm}_r}{dz} + \frac{\left[g\Gamma - (\omega_m \pm \omega_n)^2\right] |\vec{k}_m \pm \vec{k}_n|^2}{(\omega_m \pm \omega_n)^2} \widetilde{W}^{\pm}_r = 0 \\
&\sum_r \frac{d\widetilde{W}^{\pm}_r}{dz} = \frac{G^{\pm}_{mn}}{4(\omega_m \pm \omega_n)}, \quad \sum_r \widetilde{W}^{\pm}_r = -\frac{(\omega_m \pm \omega_n)\, G^{\pm}_{mn}}{4g\,|\vec{k}_m \pm \vec{k}_n|^2} \text{ at } z = 0 \\
&\sum_r \widetilde{W}^{\pm}_r = 0 \text{ at } z - H
\end{aligned}\right\} \tag{60}$$

$$\left.\begin{aligned}
&\frac{d^2 \mathring{W}^{\pm}_r}{dz^2} + \Gamma \frac{d\mathring{W}_r}{dz} + \frac{\left[g\Gamma - (\omega_m \pm \omega_n)^2\right] |\vec{k}_m \pm \vec{k}_m|^2}{(\omega_m \pm \omega_n)^2} \mathring{W}^{\pm}_r = -\frac{2\,|\vec{k}_m \pm \vec{k}_n|^2 g\,\Gamma}{(\omega_m \pm \omega_n)^3} \widetilde{W}^{\pm}_r \\
&\frac{d\mathring{W}^{\pm}_r}{dz} + \frac{g\,|\vec{k}_m \pm \vec{k}_n|^2}{(\omega_m \pm \omega_n)^2} \mathring{W}^{\pm}_r = \frac{A^{\pm}_{mnr}}{2g|\vec{k}_m \pm \vec{k}_n|^2} \frac{dW^{\pm}_r}{dz} \text{ at } z = 0 \\
&\mathring{W}^{\pm}_r = 0 \text{ at } z - H
\end{aligned}\right\} \tag{61}$$

The factor $(\omega_m \pm\omega_n)\, A^{\pm}_{mnr}\, t/(4g\,|\vec{k}_m \pm \vec{k}_n|^2)$ in (57) is dimensionless; $W^{\pm}_r\,(z)$ and $\mathring{W}^{\pm}_r\,(z)$ are velocities and $\widetilde{W}^{\pm}_r(z)$ is an acceleration.

From (57) to (61) we conclude that the primary waves create new waves of wave number $\vec{k}_m \pm \vec{k}_n$ and frequency $\omega_m \pm \omega_n$ with amplitudes depending on the driving forces $F^{\pm}_{mn}\,(z)$ and $G^{\pm}_{mn}\,(0)$. These new waves grow linearly with time if the resonance condition (56) is met. The amplitude factor $A^{\pm}_{mnr}$ is given according to (58) by

$$A^{\pm}_{mnr} = \frac{\displaystyle\int_0^H \frac{\left[g\Gamma-(\omega_m \pm \omega_n)^2\right]\bar{\rho}}{\Gamma} \quad F^{\pm}_{mn}(z)\, W^{\pm}_r(z)\, dz}{\displaystyle\int_0^H \left[g\Gamma - (\omega_m \pm \omega_n)^2\right]\bar{\rho} W^{\pm\,2}_r(z)\, dz} \tag{62}$$

If more than one internal mode is to be considered, $G^{\pm}_{mn}(0)$ has to be expanded into eigenfunctions, too, and evaluated at $z = 0$. Formulas similar to (58) and (62) will hold then for $G^{\pm}_{mn}\,(0)$.

$F^{\pm}_{mn}(z)$, $G^{\pm}_{mn}(0)$ AND $A^{\pm}_{mnr}$ IN CASE OF AN EXPONENTIALLY STRATIFIED SEA

In order to evaluate solution (57) of the secondary resonance waves, we have to know the eigenfunctions of $W^{\pm}_r(z)$, which are given by (59). They depend on the mean stratification $\bar{\rho}(z)$ and determine the amplitude factor $A^{\pm}_{mnr}$, given by (62). Equations (59) can be solved numerically for any stable stratified fluid, and it is known that the eigenvalues may vary considerably due to density changes. However, solving (59) for an exponentially stratified sea, where Γ_0= const, may be sufficient at the present time where only the magnitude of the effects is of interest.

Using $\bar{\rho}(z) = \rho_0 e^{\Gamma_0 z}$ as density distribution, W_n is given as solution of (46) to (48)

$$W_n(z) = A_n e^{-(\Gamma_0/2)\,z} \sin\, a_n(z - H) \tag{63}$$

where

$$\tan a_n H = -\frac{a_n}{\dfrac{\Gamma_0}{2} - \dfrac{gk_n^2}{\omega_n^2}} \quad , \quad a_n = \frac{1}{2}\sqrt{\frac{4(g\Gamma_0 - \omega_n^2)\, k_n^2}{\omega_n^2} - \Gamma_0^2} \tag{64}$$

and similar equations hold for $W_m(z)$. Entering equation (51) and (52), we arrive at

$$F^{\pm}_{mn} = A_m A_n e^{-\Gamma_0 z}\left\{\left[\left[\frac{|\vec{k}_m \pm \vec{k}_n|^2}{2\omega_n k_m^2}\left(g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)\right)(\mp a_n k_m^2 - a_m \vec{k}_m\cdot\vec{k}_n)\right.\right.\right.$$

$$+\frac{a_m - a_n}{2}\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^2}\right)(\omega_m \pm \omega_n)\left(a_n^2 - \frac{\Gamma_0^2}{4} \mp \frac{\omega_n}{\omega_m}\,\frac{\Gamma_0^2}{2} \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^2}\left(a_m a_n\right.\right.$$

$$\left.\left.\left.+\frac{\Gamma_0^2}{4}\right)\right)\right]\sin(a_m - a_n)(z-H) + \left[\frac{|\vec{k}_m \pm \vec{k}_n|^2}{2\omega_n k_m^2}\left(g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)\right)(\mp a_n k_m^2 + a_m \vec{k}_m\cdot\vec{k}_n)\right.$$

$$+\frac{a_m + a_n}{2}\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^2}\right)(\omega_m \pm \omega_n)\left(-a_n^2 + \frac{\Gamma_0^2}{4} \pm \frac{\omega_n}{\omega_m}\,\frac{\Gamma_0^2}{2} \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^2}\left(a_m a_n\right.\right.$$

$$\left.\left.\left.-\frac{\Gamma_0^2}{4}\right)\right)\right]\sin(a_m + a_n)(z-H) + \frac{\Gamma_0}{2}\left[|\vec{k}_m \pm \vec{k}_n|^2\left[\left(\pm 1 - \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^2}\right)\frac{g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)}{2\omega_n}\right.\right.$$

$$\left.\mp\left(\frac{g\Gamma_0}{\omega_n} - \frac{\omega_n(\omega_m \pm \omega_n)}{\omega_m}\right)\right] + (a_m - a_n)\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^2}\right)(\omega_m \pm \omega_n)\left[\mp\frac{\vec{k}_m\cdot\vec{k}_n}{2k_m^2}(a_m - a_n)\right.$$

$$\left.\left.+ a_n\left(\mp\frac{\omega_n}{\omega_m} - 1\right)\right]\right]\cos(a_m - a_n)(z-H) + \frac{\Gamma_0}{2}\left[|\vec{k}_m \pm \vec{k}_n|^2\left[\left(\mp 1 + \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^2}\right)\right.\right.$$

$$\left.\cdot\frac{g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)}{2\omega_n} \pm \left(\frac{g\Gamma_0}{\omega_n} - \frac{\omega_n(\omega_m \pm \omega_n)}{\omega_n}\right)\right] + (a_m + a_n)\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^2}\right)(\omega_m \pm \omega_n)$$

$$\left.\left.\cdot\left[\pm\frac{\vec{k}_m\cdot\vec{k}_n}{2k_m^2}(a_m + a_n) + a_n\left(\mp\frac{\omega_n}{\omega_m} - 1\right)\right]\right]\cos(a_m + a_n)(z-H)\right\} \tag{65}$$

$$G^{\pm}_{mn}(0) = \frac{1}{2}A_m A_n(\omega_m \pm \omega_n)\cdot$$

$$\cdot\left\{-\left[a_n M + \frac{\Gamma_0}{2}(a_m - a_n)N + \frac{\Gamma_0}{2}(1 + a_n)P\right]\sin(a_m - a_n)H\right.$$

$$+\left[-a_n M + \frac{\Gamma_0}{2}(a_m - a_n)N - \frac{\Gamma_0}{2}(1 + a_n)P\right]\sin(a_m + a_n)H$$

$$+\left[-\frac{\Gamma_0}{2}M + \left(a_m a_n + \frac{\Gamma_0^2}{4}\right)N + \left(a_n^2 - \frac{\Gamma_0^2}{4}\right)P\right]\cos(a_m - a_n)H$$

$$\left.+\left[\frac{\Gamma_0}{2}M + \left(a_m a_n - \frac{\Gamma_0^2}{4}\right)N - \left(a_n^2 - \frac{\Gamma_0^2}{4}\right)P\right]\cos(a_m + a_n)H\right\} \tag{66}$$

with

$$M = \pm \frac{\omega_n \Gamma_0}{\omega_m}\left(1 \pm \frac{\vec{k}_m \cdot \vec{k}_n}{k_n^2}\right) \mp \frac{g\Gamma_0 - \omega_m^2}{\omega_m^2} \; \frac{\omega_n}{g k_n^2} \left|\vec{k}_m \pm \vec{k}_n\right|^2$$

$$N = \pm \frac{\vec{k}_m \cdot \vec{k}_n}{k_m^2}\left(1 \pm \frac{\vec{k}_m \cdot \vec{k}_n}{k_n^2}\right) + \left|\vec{k}_m \pm \vec{k}_n\right|^2 \left[\mp \frac{\Gamma_0}{2} \, \frac{\omega_m \omega_n}{g k_m^2 k_n^2} + \left(\frac{\vec{k}_m \cdot \vec{k}_n}{k_n^2}\, \omega_n + \omega_m\right) \frac{1}{k_m^2(\omega_m \pm \omega_n)}\right]$$

$$P = 1 \pm \frac{\vec{k}_m \cdot \vec{k}_n}{k_n^2}$$

$A^{\pm}_{mnr}$ follows then from (62) and (65)

$$A^{\pm}_{mnr} = \frac{A_m A_n e^{-(\Gamma_0/2)H}}{\Gamma_0\left(H - \dfrac{\sin a_r H \cos a_r H}{a_r}\right)}$$

$$\left\{ \frac{Q^{\pm}\left[-\dfrac{\Gamma_0}{2} + e^{(\Gamma_0/2)H}\left(\dfrac{\Gamma_0}{2} \cos(a_m - a_n - a_r)H + (a_m - a_n - a_r) \sin(a_m - a_n - a_r) H\right)\right]}{\dfrac{\Gamma_0^2}{4} + (a_m - a_n - a_r)^2} \right.$$

$$- \frac{Q^{\pm}\left[-\dfrac{\Gamma_0}{2} + e^{(\Gamma_0/2)H}\left(\dfrac{\Gamma_0}{2} \cos(a_m - a_n + a_r)H + (a_m - a_n + a_r) \sin(a_m - a_n + a_r) H\right)\right]}{\dfrac{\Gamma_0^2}{4} + (a_m - a_n + a_r)^2}$$

$$+ \frac{R^{\pm}\left[-\dfrac{\Gamma_0}{2} + e^{(\Gamma_0/2)H}\left(\dfrac{\Gamma_0}{2} \cos(a_m + a_n - a_r)H + (a_m + a_n - a_r) \sin(a_m + a_n - a_r) H\right)\right]}{\dfrac{\Gamma_0^2}{4} + (a_m + a_n - a_r)^2}$$

$$- \frac{R^{\pm}\left[-\dfrac{\Gamma_0}{2} + e^{(\Gamma_0/2)H}\left(\dfrac{\Gamma_0}{2} \cos(a_m + a_n + a_r)H + (a_m + a_n + a_r) \sin(a_m + a_n + a_r) H\right)\right]}{\dfrac{\Gamma_0^2}{4} + (a_m + a_n + a_r)^2}$$

$$- \frac{S^{\pm} \dfrac{\Gamma_0}{2}\left[a_r - a_m + a_n + e^{(\Gamma_0/2)H}\left(\dfrac{\Gamma_0}{2} \sin(a_r - a_m + a_n)H - (a_r - a_m + a_n) \cos(a_r - a_m + a_n) H\right)\right]}{\dfrac{\Gamma_0^2}{4} + (a_r - a_m + a_n)^2}$$

$$
-\frac{S^{\pm}\,\frac{\Gamma_0}{2}\left[a_r + a_m - a_n + e^{(\Gamma_0/2)H}\left(\frac{\Gamma_0}{2}\sin(a_r + a_m - a_n)H - (a_r + a_m - a_n)\cos(a_r + a_m - a_n)H\right)\right]}{\frac{\Gamma_0^{\,2}}{4} + (a_r + a_m - a_n)^2}
$$

$$
-\frac{T^{\pm}\,\frac{\Gamma_0}{2}\left[a_r - a_m - a_n + e^{(\Gamma_0/2)H}\left(\frac{\Gamma_0}{2}\sin(a_r - a_m - a_n)H - (a_r - a_m - a_n)\cos(a_r - a_m - a_n)H\right)\right]}{\frac{\Gamma_0^{\,2}}{4} + (a_r - a_m - a_n)^2}
$$

$$
\left.-\frac{T^{\pm}\,\frac{\Gamma_0}{2}\left[a_r + a_m + a_n + e^{(\Gamma_0/2)H}\left(\frac{\Gamma_0}{2}\sin(a_r + a_m + a_n)H - (a_r + a_m + a_n)\cos(a_r + a_m + a_n)H\right)\right]}{\frac{\Gamma_0^{\,2}}{4} + (a_r + a_m + a_n)^2}\right\}
\tag{67}
$$

with

$$
\begin{aligned}
Q^{\pm} &= \frac{\left|\vec{k}_m \pm \vec{k}_n\right|^2}{2\omega_n k_m^{\,2}}\left[g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)\right]\left[\mp a_n k_m^{\,2} - a_m \vec{k}_m\cdot\vec{k}_n\right] \\
&\quad + \frac{a_m - a_n}{2}\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\right)(\omega_m \pm \omega_n)\left[a_n^{\,2} - \frac{\Gamma_0^{\,2}}{4} \mp \frac{\omega_n}{\omega_m}\,\frac{\Gamma_0^{\,2}}{2} \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^{\,2}}\left(a_m a_n + \frac{\Gamma_0^{\,2}}{4}\right)\right] \\
R^{\pm} &= \frac{\left|\vec{k}_m \pm \vec{k}_n\right|^2}{2\omega_n k_m^{\,2}}\left[g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)\right]\left[\mp a_n k_m^{\,2} + a_m \vec{k}_m\cdot\vec{k}_n\right] \\
&\quad + \frac{a_m + a_n}{2}\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\right)(\omega_m \pm \omega_n)\left[-a_n^{\,2} + \frac{\Gamma_0^{\,2}}{4} \pm \frac{\omega_n}{\omega_m}\,\frac{\Gamma_0^{\,2}}{2} \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^{\,2}}\left(a_m a_n - \frac{\Gamma_0^{\,2}}{4}\right)\right] \\
S^{\pm} &= \left|\vec{k}_m \pm \vec{k}_n\right|^2\left[\left(\pm 1 - \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^{\,2}}\right)\frac{g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)}{2\omega_n} \mp \left(\frac{g\Gamma_0}{\omega_n} - \frac{\omega_n(\omega_m \pm \omega_n)}{\omega_m}\right)\right] \\
&\quad + (a_m - a_n)\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\right)(\omega_m \pm \omega_n)\left[\mp \frac{\vec{k}_m\cdot\vec{k}_n}{2k_m^{\,2}}(a_m - a_n) + a_n\left(\mp\frac{\omega_n}{\omega_m} - 1\right)\right] \\
T^{\pm} &= \left|\vec{k}_m \pm \vec{k}_n\right|^2\left[\left(\mp 1 + \frac{\vec{k}_m\cdot\vec{k}_n}{k_m^{\,2}}\right)\frac{g\Gamma_0 \pm \omega_n(\omega_m \pm \omega_n)}{2\omega_n} \pm \left(\frac{g\Gamma_0}{\omega_n} - \frac{\omega_n(\omega_m \pm \omega_n)}{\omega_m}\right)\right] \\
&\quad + (a_m + a_n)\left(1 \pm \frac{\vec{k}_m\cdot\vec{k}_n}{k_n^{\,2}}\right)(\omega_m \pm \omega_n)\left[\pm \frac{\vec{k}_m\cdot\vec{k}_n}{2k_m^{\,2}}(a_m + a_n) + a_n\left(\mp\frac{\omega_n}{\omega_m} - 1\right)\right]
\end{aligned}
\tag{68}
$$

THE SECONDARY RESONANCE WAVE FIELD DUE TO SWELL IN CASE OF AN EXPONENTIALLY STRATIFIED SEA

With reference to observations described in the following section, we evaluated the resonance part of solution (52), using (59), (60), (66), (67), and (68). The results are based on the following numerical values:

The swell frequencies of the two primary waves are $\omega_1 = 3.808 \cdot 10^{-1}$ sec^{-1} and $\omega_2 = 3.900 \cdot 10^{-1}$ sec^{-1}. The corresponding wave numbers for a water depth of $H=$ 18 m are $k_1 = 3.00 \cdot 10^{-4}$ cm^{-1}, $k_2 = 3.08 \cdot 10^{-4}$ cm^{-1}. The density stratification is given by $\Gamma_0 = 5.811 \cdot 10^{-7}$ cm^{-1}. The corresponding periods and wavelengths of the primary waves are $\tau_1 = 16.50$ seconds, $\tau_2 = 16.11$ seconds, $\lambda_1 =$ 209.4 m, $\lambda_2 = 204.0$ m. Furthermore, for surface waves we have $a_n = i|k_n|$, $a_m = i|k_m|$. The product between the amplitudes A_n and A_m of the vertical components of the velocity is supposed to be $A_1 A_2 = 400$ cm^2 sec^{-2}. The corresponding amplitude product of the two swell waves is $\zeta_1 \zeta_2 = 889.76$ cm^2, which may be based on $\zeta_1 \approx$ 50 cm and $\zeta_2 \approx 17.8$ cm.

Because of these swell frequencies we expect an internal wave of frequency $\omega_2 - \omega_1 = 0.0092$ sec^{-1} or period $\tau_{2-1} = 11.38$ minutes. The result is shown in figure 2, which contains the amplitude of this internal wave after an interaction

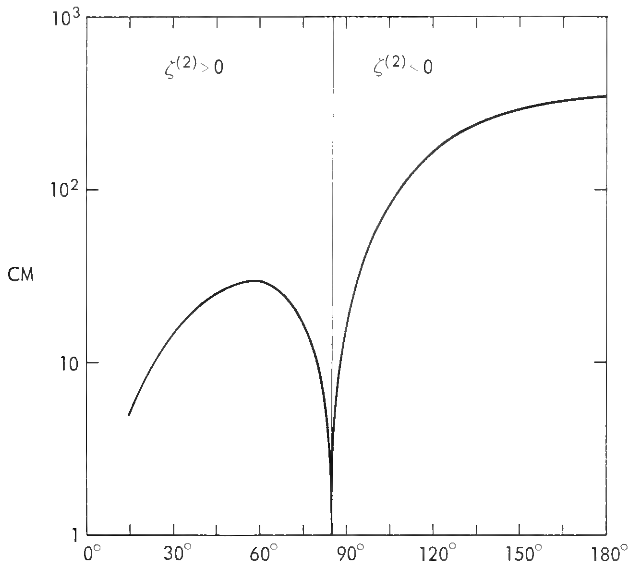

Figure 2. Amplitude (cm) of internal waves due to modulated swell after an interaction time of 10^3 seconds (16.6 minutes) as a function of the angle between the two primary waves.

time of 10^3 sec ($\approx$ 16.7 minutes). The amplitude is given in a logarithmic scale as a function of the angle θ between the two primary swell waves. Amplitudes of about 13 centimeters are obtained if the primary waves are traveling in directions containing an angle of about 60 degrees. The amplitude increases to more than 3 meters for angles of more than 150 degrees, and is only a few centimeters for angles less than 30 degrees. This seems to support the results obtained by K. E. Kenyon[11] and K. Hasselmann[12] that a narrow directional spectrum of surface waves does not lead to significant internal waves. On the other hand, the figure demonstrates that surface swell which can be described by two surface waves traveling in different directions may create internal waves very rapidly. A swell of that type would be interpreted according to equations (1) to (4) as amplitude-modulated and traveling in a direction given by $\vec{k}_1$, because the amplitude A_1 is much larger than A_2 and therefore would govern the swell.

MEASUREMENTS ON SWELL AND INTERNAL WAVES

In order to test the theory, simultaneous measurements on surface swell and internal waves were made in October 1966 at the NEL Tower, running from 3 October, 1130 to 6 October, 1215. Two wave-height sensors were used to derive the directional spectrum of the swell – one fixed at the NEL Tower, the second at a distance of 39.3 meters toward the west. The voltage output was recorded on punch tape. The temperature fluctuations were measured by means of two vertical thermistor arrays. The spacing between the thermistors was 75 centimeters, reaching from the bottom up to 2.75 meters below mean sea surface. The two arrays were located in positions along a line from southwest to northeast. The distance between the arrays was 284.4 meters. The temperature fluctuations were recorded on an analog recorder and on punch tape. Simultaneous records of swell and temperature fluctuations were obtained. The complete analysis of these data will be published later. Preliminary results support the theory given above. Figure 3 shows the power spectra of the two wave-height sensors for data from 4 October, 1230–1330. The voltage output of the two sensors has been used directly for these calculations. This output is different for both, and this is the reason for spectrum 2 showing higher intensities than spectrum 1. Otherwise, both spectra coincide quite well. The phase difference changes linearly with frequency as is to be expected. But there occurs a remarkable hole at frequencies of about 0.061 cycle per second, corresponding to $\omega \approx 0.38$ sec^{-1} or $\tau \approx 16.4$ sec. The energy decreases in both spectra from about 12 to 1.2 (arbitrary units), and the phase differences change from 35° to –70°, which indicates that the swell in this spectral range consists of two waves, one with $\omega_2 = 3.9 \times 10^{-1}$ sec^{-1} traveling in direction 30° and a second one with $\omega_1 = 3.808 \times 10^{-1}$ sec^{-1} traveling in a direction of about 270°. The angle between both is about 120°. From figure 2 it follows that within a quarter of an hour these two swell waves would create an internal wave of $\tau = 11.4$ minutes with an amplitude of 1.5 meters

The swell in the vicinity of the NEL Tower therefore seems to be modulated as indicated by figure 3. If the stratification is favorable, that is, if $|\vec{k}_m - \vec{k}_n|^2 \cdot (\omega_m - \omega_n)^2$ is an eigenvalue, a very intensive energy transfer from the swell

frequencies ω_m and ω_n to internal waves of the first mode occurs, producing internal waves of amplitudes of several meters. These internal waves have the same characteristics as the swell modulation. Their period coincides with that of the modulation, and the wavelength is the same. Additionally, they travel in the same direction as the modulation.

The agreement between internal wave periods and the periods of the swell modulations was tested. Data from the wave-height sensors for the time from 4 October 1966, 1900 to 5 October 1966, 0145, were used to determine the periods of the modulations. The internal wave periods were taken from the analog records for the time 3 October, 1500 to 5 October, 0300. Histograms of both

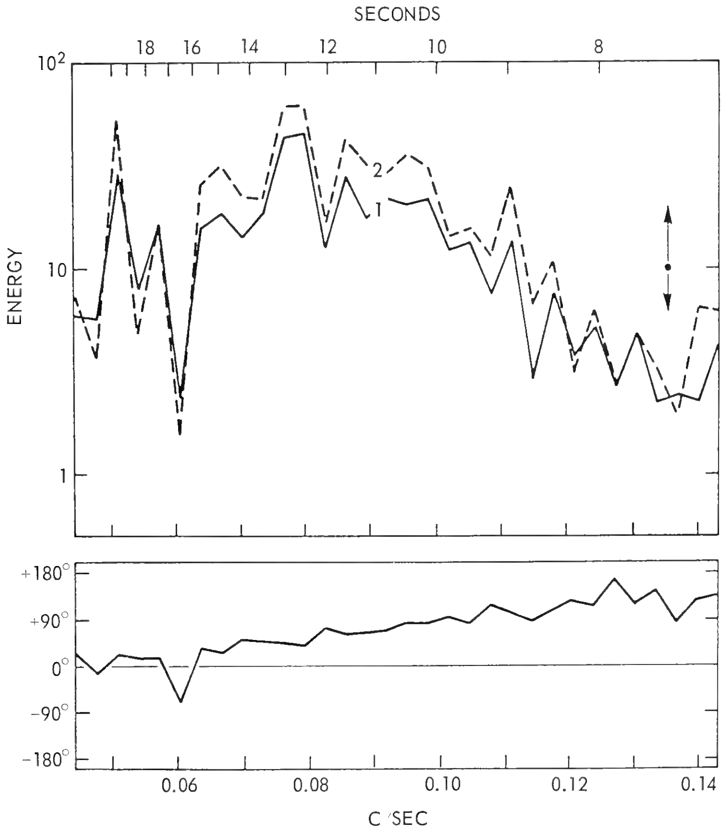

Figure 3. Swell spectra of wave height sensors 1 and 2 at positions near the NEL Tower, and phase difference between both records (4 October 1966, 1230-1330; sampling rate, 1 second); the arrow indicates the 95-percent confidence limit.

periods are shown in figure 4. They show the number of occurrences of periods between 2 and 22 minutes in percent. The number of waves contained in each histogram is about 120. The figure is in good agreement with the theory given above, both distributions are peaked in the period range of 4-to-10 minutes and decrease toward periods larger than 20 minutes. Final conclusions, however, are possible only after the spectral analysis of the entire data. A better agreement between these histograms cannot be expected because the internal waves are strongly dependent on the changing local mean stratification,[5] whereas the swell is governed by quite different sources.

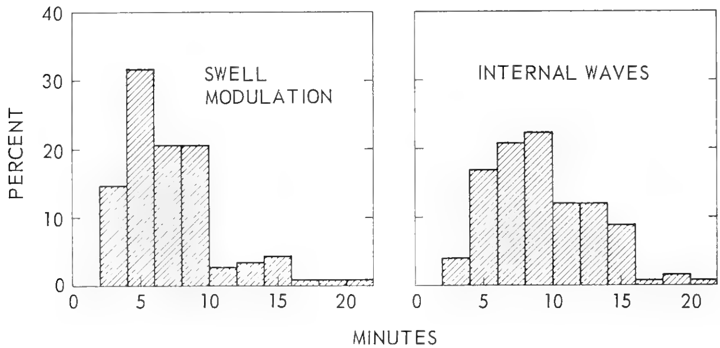

Figure 4. Histograms of periods of swell modulations (left) and of internal waves (right) at the NEL Tower.

CONCLUSIONS

1. Resonant interaction between amplitude-modulated swell and internal waves may create internal waves of 2-to-20-minute periods. Internal waves with amplitudes of 1-to-3 meters can be produced within an interaction time of 15 minutes if the stratification of the water is such that

$$\left|\vec{k}_1 - \vec{k}_2\right|^2 / (\omega_1 - \omega_2)^2$$

is an eigenvalue of the internal wave equation ($\vec{k}_1$, $\vec{k}_2$, ω_1, and ω_2 being wave numbers and frequencies of the modulated swell).

2. Internal waves due to amplitude-modulated swell have the same characteristics as the modulation. Specifically, they have the same wavelength and period and travel in the same direction as the amplitude modulation.

3. The resonance process is most efficient in the case of a modulation which travels in the same direction as the carrier wave (the main constituent) of a swell. The creation of internal waves of this type is strongly dependent on the stratification of the water.

4. Preliminary evaluations of temperature and swell records from the NEL Tower show good agreement between periods of swell modulation and internal wave periods.

5. It is likely that internal waves are created on the entire continental shelf off Southern California during times of favorable stratification. This depends on tide and wind.

RECOMMENDATIONS

1. Verify, with additional measurements at the NEL Tower, the theory that internal waves are produced by swell.

2. Acquire, for complete analysis, long records of temperature fluctuations and surface waves in order to compute spectra with high resolution and statistical confidence.

3. Check, by measurements of surface waves at two positions, and slick observations, the result that internal waves travel in the direction of the swell modulation.

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13. ABSTRACT

The relationship of internal waves, with 2-to-20-minute periods, to surface waves was investigated. It was found that (1) the interaction between amplitude-modulated swell and internal waves may create internal waves of 2-to-20-minute periods; (2) internal waves due to amplitude-modulated swell have the same characteristics as the modulation; and (3) the resonance process is most efficient in the case of a modulation traveling in the same direction as the carrier wave of a swell.

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	ROLE	WT	ROLE	WT	ROLE	WT
Internal Waves - Shallow Water						
Ocean Waves - Analysis						

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